February 2025 Problem Submission
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This is Banach's fixed-point theorem—the following is an expansion and rearrangement of Sutherland's proof of such [1].

Let $\langle x_n \rangle$ be a sequence such that $x_1 \in S$ is an arbitrary element, and $x_n = f(x_{n-1})$ for $n \in \mathbf{N}^+$ such that $n > 1$. We first establish that $\langle x_n \rangle$ converges to some $p \in S$.

Lemma 1. For $m, n \in \mathbf{N}^+$ such that $m > n$, we have

$$d(x_m, x_n) \leq \sum_{j=n}^{m-1} d(x_{j+1}, x_j)$$

Proof. Fix $n \in \mathbf{N}^+$. For $\ell \in \mathbf{N}^+$, let $P(\ell)$ be the proposition

$$d(x_{n+\ell}, x_n) \leq \sum_{j=n}^{n+\ell-1} d(x_{j+1}, x_j)$$

We proceed inductively.

Base Case Let $\ell = 1$. Then,

$$d(x_{n+1}, x_n) = \sum_{j=n}^{n} d(x_{j+1}, x_j)$$

Hence, we have that $P(1)$ holds.

Inductive Step Suppose there exists $k \in \mathbf{N}^+$ such that

$$d(x_{n+k}, x_n) \leq \sum_{j=n}^{n+k-1} d(x_{j+1}, x_j)$$

Then,

$$\begin{aligned} d(x_{n+k+1}, x_n) &\leq d(x_{n+k+1}, x_{n+k}) + d(x_{n+k}, x_n) \\ &\leq d(x_{n+k+1}, x_{n+k}) + \sum_{j=n}^{n+k-1} d(x_{j+1}, x_j) \\ &= \sum_{j=n}^{n+k} d(x_{j+1}, x_j) \end{aligned}$$

Hence, we have that $P(k)$ implies $P(k+1)$.

By the principle of mathematical induction, it's clear that $P(\ell)$ holds for all $\ell \in \mathbf{N}^+$; further, since n was arbitrary, by universal introduction[1], the inequality holds for all $n \in \mathbf{N}^+$ as well.

We arrive at the formulation in the lemma by substituting m for $n + \ell$. $\square$

[1] See [2], [3, Section 10.2], and [4, Section 3].

Lemma 2. For $n \in \mathbf{N}^+$ such that $n > 1$, we have

$$d(x_{n+1}, x_n) \leq \mu^{n-1} d(x_2, x_1)$$

Proof. For $n \in \mathbf{N}^+$ such that $n > 1$, let $P(n)$ be the proposition

$$d(x_{n+1}, x_n) \leq \mu^{n-1} d(x_2, x_1)$$

We proceed inductively.

<u>Base Case</u> Let $n = 2$. Then,

$$d(x_3, x_2) = d(f(x_2), f(x_1)) \leq \mu\, d(x_2, x_1) = \mu^{2-1}\, d(x_2, x_1)$$

Hence, we have that $P(2)$ holds.

<u>Inductive Step</u> Suppose there exists $k \in \mathbf{N}^+$, $k > 1$, such that

$$d(x_{k+1}, x_k) \leq \mu^{k-1}\, d(x_2, x_1)$$

Then,

$$\begin{aligned} d(x_{k+2}, x_{k+1}) &= d(f(x_{k+1}), f(x_k)) \\ &\leq \mu\, d(x_{k+1}, x_k) \\ &\leq \mu[\mu^{k-1}\, d(x_2, x_1)] \\ &= \mu^k\, d(x_2, x_1) \end{aligned}$$

Hence, we have that $P(k)$ implies $P(k+1)$.

By the principle of mathematical induction, it's clear that $P(n)$ holds for all $n \in \mathbf{N}^+$ such that $n > 1$. $\square$

Let $m, n \in \mathbf{N}^+$ such that $m > n$. Using **Lemma 1**, **Lemma 2**, and the closed-form formula for the sum of a geometric series, see that

$$\begin{aligned}
d(x_m, x_n) &\leq \sum_{j=n}^{m-1} d(x_{j+1}, x_j) \\
&\leq \sum_{j=n}^{m-1} \mu^{j-1}\, d(x_2, x_1) \\
&= \left(\sum_{j=n}^{m-1} \mu^{n-1} \mu^{j-n} \right) d(x_2, x_1) \\
&= \mu^{n-1} \left(\sum_{q=0}^{m-n-1} \mu^q \right) d(x_2, x_1) \\
&= \mu^{n-1} \left(\frac{1-\mu^{m-n}}{1-\mu} \right) d(x_2, x_1) \\
&< \frac{\mu^{n-1}}{1-\mu} d(x_2, x_1)
\end{aligned}$$

Note that $0 \leq \mu < 1$ means $\frac{\mu^{n-1}}{1-\mu} d(x_2, x_1)$ is a geometric sequence which converges to 0 as n approaches infinity. I.e., there exists $N \in \mathbf{N}^+$ such that for all $\epsilon \in \mathbf{R}^+$, if $n > N$ then $\frac{\mu^{n-1}}{1-\mu} d(x_2, x_1) < \epsilon$. Take said N. Given $\epsilon \in \mathbf{R}^+$, we can choose $m, n \in \mathbf{N}^+$ such that $m > n$ and $n > N$, yielding

$$d(x_m, x_n) < \frac{\mu^{n-1}}{1-\mu} d(x_2, x_1) < \epsilon$$

Hence, $\langle x_n \rangle$ is Cauchy.

Since (S, d) is complete and $\langle x_n \rangle$ is Cauchy, we have that $\langle x_n \rangle$ converges to some $p \in S$.

As a penultimate result, see that f is uniformly continuous; for if we're given $\epsilon \in \mathbf{R}^+$, we can simply set $\delta = \frac{\epsilon}{\mu}$. Take $a, b \in S$. Then,

$$d(a, b) < \delta \implies d(f(a), f(b)) \leq \mu \, d(a, b) < \mu \left(\frac{\epsilon}{\mu} \right) = \epsilon$$

Synthesizing $\langle x_n \rangle$'s converging to some $p \in S$ with f's being uniformly continuous, we have

$$f(p) = f(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_{n+1} = p$$

Thus, there exists $p \in S$ such that $f(p) = p$. $\square$

References

[1] Sutherland, W. A. (2009b). Introduction to metric and topological spaces. Oxford: Oxford University Press.

[2] Shapiro, S., & Kouri Kissel, T. (2022, June 29). Classical logic. Stanford Encyclopedia of Philosophy. Stanford University. `https://plato.stanford.edu/entries/logic-classical/`. Accessed 4 February 2025

[3] (N.d.). Introduction to logic. Stanford University. `http://intrologic.stanford.edu/chapters/chapter\_10.html`. Accessed 4 February 2025

[4] Biderman, S. (2018, August 23). Proof by induction involving two variables. Mathematics Stack Exchange. `https://math.stackexchange.com/questions/2891651/proof-by-induction-involving-two-variables`. Accessed 4 February 2025